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On the isoperimetric number of a k-degree Cayley graph☆

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ABSTRACT

In this work, we shall concentrate on the isoperimetric properties of the k-degree Cayley graphs $G_{k,n}$, which were proposed recently for building interconnection networks. We shall give the exact isoperimetric number $i(G_{k,n})$ when $n = 2$, and an upper bound of $i(G_{k,n})$ in the general case.

1. Introduction

Given a graph G and a subset $S \subset V(G)$ satisfying $|S| \leqslant \frac{|V(G)|}{2}$, let ∂S denote the *edge-boundary* of S, i.e.

$$\partial S = \{\{u, v\} \in E(G) | u \in S, v \in V(G) \setminus S\}.$$

The quotient $i_S(G) = \frac{|\partial S|}{|S|}$ is called the *isoperimetric quotient* related to S. The *isoperimetric number* of G is defined as

$$i(G) = \min_{1 \leqslant |S| \leqslant \frac{|V(G)|}{2}} i_S(G). \tag{1.1}$$

A subset S of vertices which achieves the minimum value of (1.1) is called an *isoperimetric set* of G. We refer the reader to [1] or [2] for a discussion of the basic results and various interesting properties of $i(G)$ and to [3] for a comprehensive survey of this and related problems.

The isoperimetric number is of interest to combinatorialists for several reasons. One is that bounds on the eigenvalue spectrum of a graph can be obtained from it. In particular, Mohar, in [1], presented the close relations between $i(G)$ and the algebraic connectivity λ, which is the second-smallest eigenvalue of the Laplacian matrix of G. The isoperimetric number $i(G)$ can also be viewed as a measure of the connectedness of the graph G and is therefore relevant to the problem of constructing good expanders.

On the other hand, the properties of the isoperimetric set are very important in applications such as graph partitioning, parallel computation, randomized algorithms etc. The recent papers of Leo Grady and Eric L. Schwartz [4,5] concern graph partitioning and related algorithms in image segmentation. When applied to parallel computation, the isoperimetric set can be viewed as the computing tasks that may be mapped to the same machine so that the communication loads between machines would be minimized.

There are some related works on the isoperimetric properties of graphs, such as [6–8].

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The design of interconnection networks is an important issue in parallel processing or distributed systems, and many networks have been proposed in the literature. The authors of [9] introduced a new family of Cayley graphs, called k-degree Cayley graphs, for building interconnection networks. The k-degree Cayley graph possesses many valuable topological properties, such as regularity with degree k, logarithmic diameter, and maximal fault tolerance. They also presented an optimal shortest path routing algorithm for the k-degree Cayley graph.

In this work, we concentrate on the isoperimetric properties of the k-degree Cayley graphs.

In Section 2, we review the definition and properties of the k-degree Cayley graph, and give some terminology and properties which are important. In Section 3, by means of several lemmas, we present the exact isoperimetric number of the k-degree Cayley graph with $n = 2$. Additionally, we give an upper bound of the isoperimetric number of the k-degree Cayley graph, from structure analysis. In the final section, we indicate future work on this problem.

2. Definitions and basic properties

Definition 2.1. Let H be a finite group and let Ω be a generating set for H. Assume that $1 \notin \Omega$ and that $\Omega = \Omega^{-1}$. We define the Cayley graph for H with respect to Ω, denoted as $G(H, \Omega)$, as follows: the vertices of G are the elements of H. Vertex v_1 is connected to vertex v_2 if and only if $v_1 = v_2\omega$ for some $\omega \in \Omega$.

And the following is a formal definition of the k-degree Cayley graph introduced in [9]:

Definition 2.2. A k-degree Cayley graph $G_{k,n}$ is an undirected graph with $n(k-1)^n$ vertices for any integers $n \geqslant 2$ and $k \geqslant 3$. Each vertex v of $G_{k,n}$ has the form $s_0s_1\cdots s_{m-1}\tilde{s}_m s_{m+1}\cdots s_{n-1}$ corresponding to a string of n symbols selected from $\{0, 1, \ldots, k-2\}$ such that exactly one symbol $\tilde{s}_m$ is in marked form and the others are in unmarked form. We sometimes use v^m to represent a vertex v with the marked symbol on position m; thus, the symbols v^m and v are used interchangeably throughout this article. Let $s_i^* = s_i$ or $\tilde{s}_i$. Each edge is of type $(v, \delta(v))$, where $\delta \in \{f, f^{-1}, g_1, g_2, \ldots, g_{k-2}\}$ is a generator defined as follows:

1. $f(u^m) = v^{(m-1) \bmod n}$, where $u^m = s_0^*s_1^*\cdots s_{n-1}^*$, $v^{(m-1) \bmod n} = s_1^*s_2^*\cdots s_{n-1}^*\alpha^*$ and $\alpha = (s_0+1) \bmod (k-1)$;
2. $f^{-1}(u^m) = v^{(m+1) \bmod n}$, where $u^m = s_0^*s_1^*\cdots s_{n-1}^*$, $v^{(m+1) \bmod n} = \beta^*s_0^*s_1^*\cdots s_{n-2}^*$ and $\beta = (s_{n-1}-1) \bmod (k-1)$;
3. $g_i(u^m) = v^m$, where $u^m = s_0^*s_1^*\cdots s_{n-1}^*$, $v^m = s_0^*s_1^*\cdots s_{n-2}^*\gamma^*$ and $\gamma = (s_{n-1}+i) \bmod (k-1)$ for $1 \leqslant i \leqslant k-2$.

Thus let finite group $H = \{u^m \mid 0 \le m \le n-1\}$, where u^m is as the above definition, and let a generating set for H be $\Omega = \{f, f^{-1}, g_1, g_2, \ldots, g_{k-2}\}$, where $f,\ f^{-1},\ g_i\ (1 \le i \le k-2)$ follow the description in the above definition, one may easily see that a k-degree Cayley graph $G_{k,n}$ is a particular Cayley graph.

Proposition 2.3 ([9]). *$G_{k,n}$, $n \geqslant 2$ and $k \geqslant 3$, is a regular graph of degree k with $\frac{kn(k-1)^n}{2}$ edges.*

For any vertex $v^m = s_0^*s_1^*\cdots s_{n-2}^*s_{n-1}^* \in V(G_{k,n})$, the *sub-clique* $Q^m_{s_0s_1\cdots s_{n-2}}$ is the subgraph induced by the set $\{v^m, g_1(v^m), \ldots, g_{k-2}(v^m)\}$, which obviously is a complete graph K_{k-1}. This sub-clique can be uniquely located by its *level m* (the position in marked form) and *prefix* $s_0s_1\cdots s_{n-2}$ (the symbol sequence changes in the sub-clique). We use the notation $L(Q_i)$ or $L(v)$ to take the level of the sub-clique Q_i or the vertex $v \in V(G_{k,n})$ respectively.

Proposition 2.4. *For any two different vertices u, v in the same sub-clique Q, $f(u)$ and $f(v)$ belong to different sub-cliques. The same is true for f^{-1}.*

Proof. Suppose that $u = s_0^*s_1^*\cdots s_{n-2}^*x^*$ and $v = s_0^*s_1^*\cdots s_{n-2}^*y^*$ such that $x \neq y$. Then

$$f(u) = s_1^*\cdots s_{n-2}^*x^*(s_0+1)^*, \qquad f(v) = s_1^*\cdots s_{n-2}^*y^*(s_0+1)^*$$

The prefixes of $f(u)$ and $f(v)$ are different, so they must belong to different sub-cliques. The same argument can apply to f^{-1}. □

Given a graph $G_{k,n}$ and a subset S of $V(G_{k,n})$, for any sub-clique Q, the *cover index* $c_S(Q)$ is defined as

$$c_S(Q) = |S \cap V(Q)|.$$

Clearly, we have $0 \leqslant c_S(Q) \leqslant k-1$. The sub-clique Q is *fully covered* by S if $c_S(Q) = k-1$, and *uncovered* if $c_S(Q) = 0$. If $1 \leqslant c_S(Q) \leqslant k-2$, we say Q is *partially covered* by S. The *partially covered index* $p(S)$ of S is defined as

$$p(S) = \#\left\{Q \mid 1 \leqslant c_S(Q) \leqslant k-2\right\},$$

where $\#A$ denotes the number of elements in set A.

Definition 2.5. Given a graph $G_{k,n}$, the contraction graph $G^*_{k,n}$ is a simple graph defined as follows:

$$V(G^*_{k,n}) = \left\{\text{All the sub-cliques } Q \text{ of } G_{k,n}\right\}$$
$$E(G^*_{k,n}) = \left\{\{Q_i, Q_j\} \mid \exists\, u \in Q_i, \exists\, v \in Q_j, \{u, v\} \in E(G_{k,n})\right\}.$$

We can easily see that, if $\{Q_i, Q_j\} \in E(G^*_{k,n})$, then $L(Q_i) - L(Q_j) \equiv \pm 1 (\bmod\ n)$.

Then we have the following proposition:

Proposition 2.6.

$$i(G_{k,n}) \leqslant \begin{cases} \dfrac{1}{k-1} \cdot i(G^*_{k,n}), & n \geqslant 3 \\ \dfrac{2}{k-1} \cdot i(G^*_{k,n}), & n = 2. \end{cases} \tag{2.1}$$

Moreover, if $G_{k,n}$ has an isoperimetric set S such that $p(S) = 0$, then the equality holds.

Proof. We prove (2.1) in two cases.

Case 1. $n \geqslant 3$. Straightforwardly from the definitions above, $G^*_{k,n}$ is a regular graph of degree $2(k-1)$ with $n(k-1)^n$ edges. On the other hand, there are $\frac{kn(k-1)^n}{2} - n(k-1)^{n-1}\binom{k-1}{2} = n(k-1)^n$ edges connecting different sub-cliques in $G_{k,n}$. There is a bijection between these two edge sets.

Suppose S^* is an isoperimetric set of $G^*_{k,n}$ and S is the corresponding set in $G_{k,n}$. Then $|S| = (k-1)|S^*|$. And $|\partial S| = |\partial S^*|$. Therefore,

$$i(G_{k,n}) \leqslant i_S(G_{k,n}) = \frac{|\partial S|}{|S|} = \frac{|\partial S^*|}{(k-1)|S^*|} = \frac{1}{k-1} \cdot i(G^*_{k,n}).$$

Case 2. $n = 2$. Given a sub-clique Q, without loss of generality, we assume that $L(Q) = 0$. For $\forall\, v \in V(Q)$, $L(f(v)) = L(f^{-1}(v)) = 1$ since $1 \equiv -1 (\mathrm{mod}\ 2)$. According to Proposition 2.4, we know that, for each edge in $G^*_{k,2}$, there are two corresponding edges in $G_{k,2}$, since there are exactly $k-1$ sub-cliques in each level.

Then for any isoperimetric set S^* of $G^*_{k,2}$, assume S is the corresponding set in $G_{k,2}$. $|S| = (k-1)|S^*|$ and $|\partial S| = 2|\partial S^*|$. Therefore,

$$i(G_{k,2}) \leqslant i_S(G_{k,2}) = \frac{|\partial S|}{|S|} = \frac{2|\partial S^*|}{(k-1)|S^*|} = \frac{2}{k-1} \cdot i(G^*_{k,2}).$$

If S is an isoperimetric set of $G_{k,n}$ with $p(S) = 0$, and S^* is the corresponding set in $G^*_{k,n}$, the equality holds if and only if S^* is an isoperimetric set of $G^*_{k,n}$. Suppose S^* is not the isoperimetric set of $G^*_{k,n}$. If S^*_1 is an isoperimetric set of $G^*_{k,n}$ and S_1 is the corresponding set in $G_{k,n}$, then clearly we have

$$i_{S^*_1}(G^*_{k,n}) < i_{S^*}(G^*_{k,n})$$

$$i_S(G_{k,n}) = \begin{cases} \dfrac{1}{k-1} \cdot i_{S^*}(G^*_{k,n}), & n \geqslant 3 \\ \dfrac{2}{k-1} \cdot i_{S^*}(G^*_{k,n}), & n = 2 \end{cases}$$

$$i_{S_1}(G_{k,n}) = \begin{cases} \dfrac{1}{k-1} \cdot i_{S^*_1}(G^*_{k,n}), & n \geqslant 3 \\ \dfrac{2}{k-1} \cdot i_{S^*_1}(G^*_{k,n}), & n = 2 \end{cases}$$

which means $i_{S_1}(G_{k,n}) < i_S(G_{k,n})$, contradicting that S is an isoperimetric set of $G_{k,n}$. □

From Proposition 2.6, we can see that, if there exists an isoperimetric set S with $p(S) = 0$ for any $G_{k,n}$, then the equality of (2.1) holds, and we just need to investigate the contraction graph $G^*_{k,n}$.

3. Isoperimetric number of $G_{k,n}$

First, let us discuss the case $n = 2$. We need the following lemmas.

Lemma 3.1. *$G^*_{k,2}$ is a complete bipartite graph with equal partition size $k-1$.*

This lemma is straightforward according to the definition of the k-degree Cayley graph and its contraction graph and the proof of Proposition 2.6.

Lemma 3.2 ([1]).

$$i(K_{m,m}) = \begin{cases} \dfrac{m}{2}, & m \text{ is even;} \\ \dfrac{m^2+1}{2m}, & m \text{ is odd.} \end{cases}$$

Lemma 3.3.

$$i(G_{k,2}) \leqslant \begin{cases} 1, & k \text{ is odd,} \\ \dfrac{(k-1)^2+1}{(k-1)^2}, & k \text{ is even.} \end{cases} \tag{3.1}$$

Proof.

$$i(G_{k,n}) \leqslant \frac{2}{k-1} \cdot i(G^*_{k,n}) = \frac{2}{k-1} \cdot i(K_{k-1,k-1}) = \begin{cases} 1, & k \text{ is odd,} \\ \dfrac{(k-1)^2+1}{(k-1)^2}, & k \text{ is even.} \end{cases} \tag{3.2}$$

In (3.2), we directly use Proposition 2.6, Lemmas 3.1 and 3.2 successively. □

The equality of (3.1) holds if we can find an isoperimetric set S of $G_{k,2}$ with $p(S) = 0$. In order to do this, we need the following lemmas.

Lemma 3.4. *If S is an isoperimetric set of $G_{k,2}$ with minimum $p(S) > 0$, and Q is a partially covered sub-clique, then:*
(1) $c_S(Q) \geqslant \frac{k-2}{2}$;
(2) $c_S(Q)$ for all partially covered Q s are equal;
(3) if $c_S(Q) \geqslant \frac{k}{2}$, then $|S| = \frac{|V(G_{k,2})|}{2} = (k-1)^2$.

Proof. (1) If there exists a partially covered sub-clique Q such that $1 \leqslant c_S(Q) \leqslant \frac{k-3}{2}$, then let $S^* = S - v$ for some vertex $v \in S \cap V(Q)$; then we have $|S^*| = |S| - 1$ and

$$|\partial S^*| \leqslant |\partial S| - (k-1-c_S(Q)) + c_S(Q) - 1 + 2 \leqslant \begin{cases} |\partial S| - 1, & k \text{ is odd,} \\ |\partial S| - 2, & k \text{ is even.} \end{cases}$$

According to Lemma 3.3, we have

$$i_{S^*}(G_{k,2}) = \frac{|\partial S^*|}{|S^*|} \leqslant \frac{|\partial S|}{|S|} = i(G_{k,2}).$$

We can continue doing this until we find a set S' with either smaller isoperimetric quotient $i_{S'}(G_{k,2}) < i_S(G_{k,2})$, which means S is not an isoperimetric set of $G_{k,2}$, or smaller partially covered index $p(S') < p(S)$, which means $p(S)$ is not the minimum, contradicting the properties of S.

(2) Suppose there exist two partially covered sub-cliques Q_i and Q_j satisfying $c_S(Q_i) \neq c_S(Q_j)$. Without loss of generality, assume that $c_S(Q_i) = x_1 > x_2 = c_S(Q_j)$.

Let $R \subset S \cap V(Q_j)$ and $A = V(Q_i) \setminus S$ such that $|R| = |A| = k - 1 - x_1$. This can be done since $x_1 + x_2 \geqslant 1 + 2x_2 \geqslant k - 1$ according to Lemma 3.4(1). And let $S' = (S \cup A) \setminus R$. Then we have $|S'| = |S|$ and

$$\begin{aligned} |\partial S'| &\leqslant |\partial S| - x_1(k-1-x_1) + 2(k-1-x_1) - (k-1-x_1)(k-1-x_2) \\ &\quad + (x_2 - (k-1-x_1))(k-1-x_1) + 2(k-1-x_1) \\ &= |\partial S| + (k-1-x_1)(2x_2+6-2k) \leqslant |\partial S|. \end{aligned}$$

The last inequality holds because $x_2 < x_1 \leqslant k - 2$. Therefore,

$$i_{S'}(G_{k,2}) = \frac{|\partial S'|}{|S'|} \leqslant \frac{|\partial S|}{|S|} = i(G_{k,2}).$$

And $p(S') = p(S) - 1$, a contradiction with $p(S)$ minimum.

(3) If $|S| < (k-1)^2$, then let $S' = S + v$ for some vertex $v \in V(Q) \setminus S$. We have $|S'| = |S| + 1 \leqslant (k-1)^2 = \frac{|V(G_{k,2})|}{2}$ and $|\partial S'| \leqslant |\partial S| - c_S(Q) + (k-1-c_S(Q)-1) + 2 = |\partial S| - 2c_S(Q) + k \leqslant |\partial S|$. Then we have

$$i_{S'}(G_{k,2}) = \frac{|\partial S'|}{|S'|} \leqslant \frac{|\partial S|}{|S|+1} < i(G_{k,2}).$$

This is a contradiction to S being an isoperimetric set of $G_{k,2}$. □

Lemma 3.5. *For any sub-clique Q in $G_{k,2}$, if $v_1, v_2 \in V(Q)$ and $v_1 \neq v_2$, then the four vertices $f(v_1), f^{-1}(v_1), f(v_2), f^{-1}(v_2)$ locate in at least three different sub-cliques.*

Proof. Suppose that $v_1 = s_0^* s_1^* \cdots s_{n-2}^* x^*$ and $v_2 = s_0^* s_1^* \cdots s_{n-2}^* y^*$ such that $x \neq y$. By Proposition 2.4, we know that $f(v_1) = s_1^* \cdots s_{n-2}^* x^* (s_0+1)^*$ and $f(v_2) = s_1^* \cdots s_{n-2}^* y^* (s_0+1)^*$ locate in different sub-cliques. Moreover,

$$f^{-1}(v_1) = (x-1)^* s_0^* s_1^* \cdots s_{n-2}^*, \qquad f^{-1}(v_2) = (y-1)^* s_0^* s_1^* \cdots s_{n-2}^*.$$

At most one of $(x-1) \equiv s_1 \bmod n$ and $(y-1) \equiv s_1 \bmod n$ can hold, since $x \neq y$, which means that at least one of $f^{-1}(v_1)$ and $f^{-1}(v_2)$ locates in another sub-clique. And the proof is complete. □

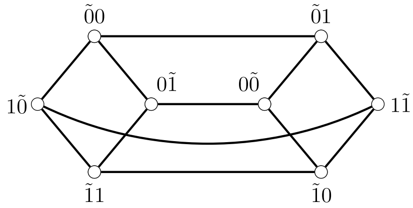


Fig. 1. An example of a degree 3 Cayley graph.

Lemma 3.6 ([1]). *The n-dimensional cube graph $Q_n = K_2^n$ has $i(Q_n) = 1$.*

Lemma 3.7. *If S is an isoperimetric set of $G_{k,2}$ with minimum $p(S)$, then $p(S) = 0$.*

Proof. For $k = 3$, we can see, from Fig. 1, that $G_{3,2}$ is 3-cube Q_3. By Lemma 3.6, we know $i(Q_3) = 1$. We can choose $S = \{\tilde{0}0, \tilde{0}1, 0\tilde{1}, 0\tilde{0}\}$ as its isoperimetric set, and $p(S) = 0$, which is minimum. So in the following, we can consider just $k \geqslant 4$.

Suppose S is an isoperimetric set in $G_{k,2}$ with minimum $p(S) > 0$. Then all the properties of Lemmas 3.4 and 3.5 are satisfied. Let m denote the number of uncovered vertices in each partially covered sub-clique Q, i.e. $m = k - 1 - c_S(Q)$. Clearly we have $1 \leqslant m \leqslant \frac{k}{2}$.

Case 1. $|S| = \frac{|V(G_{k,2})|}{2} = (k-1)^2$.

There are at least $\frac{[m,k-1]}{m}$ ($[a, b]$ denotes the least common multiple of a and b) sub-cliques partially covered by S. Select $\frac{[m,k-1]}{m}$ sub-cliques among these partially covered sub-cliques and denote them by Q_i ($1 \le i \le \frac{[m,k-1]}{m}$). Then let $S' = S \cup V(Q_i) \setminus V(Q_j)$, where $1 \le i \le \frac{[m,k-1]}{m} - \frac{[m,k-1]}{k-1}$ and $1 \le j \le \frac{[m,k-1]}{k-1}$. We can find that $|S'| = |S|$ and

$$|\partial S'| \leqslant |\partial S| - \frac{[m,k-1]}{m} \cdot (k-1-m)m + \left(\frac{[m,k-1]}{m} - \frac{[m,k-1]}{k-1}\right) \cdot 2m + \frac{[m,k-1]}{k-1} \cdot 2(k-1-m)$$
$$= |\partial S| - \frac{(k-5)(k-1-m)}{k-1} \cdot [m, k-1]. \tag{3.3}$$

Clearly, we have $|\partial S'| \leqslant |\partial S|$ when $k \geqslant 5$. S' is a set with either $i_{S'}(G_{k,2}) < i_S(G_{k,2})$ or $p(S') < p(S)$, a contradiction. Only $k = 4$ remains to be discussed in the following.

For the isoperimetric set S with minimum $p(S) > 0$ and $|S| = \frac{|V(G_{k,2})|}{2}$, and Q a partially covered sub-clique, we claim the following two properties:

(i) For any vertex $u \in V(Q) \setminus S$, we must have $f(u) \notin S$ and $f^{-1}(u) \notin S$. Otherwise, following the construction of S' above, we can choose this Q to be covered by S', and this will give minus at least 2 in the right of (3.3) and make $|\partial S'| \leqslant |\partial S|$.

(ii) For any vertex $v \in V(Q) \cap S$, we must have $f(v) \in S$ and $f^{-1}(v) \in S$. Otherwise, following the construction of S' above, we can choose this Q to be uncovered by S', and this will give minus at least 2 in the right of (3.3) and make $|\partial S'| \leqslant |\partial S|$.

When $k = 4$, each sub-clique is a triangle and altogether we have six triangles.

If $m = 1$, we have three triangles partially covered, one triangle fully covered and two triangles uncovered. Without loss of generality, assume that Q_1 is a partially covered triangle with level 0. Let $v_1, v_2 \in V(Q_1) \cap S$; then all three triangles of level 1 are at least partially covered. This can be verified easily via Lemma 3.5 and the claim (ii) above. Conversely, let Q_2 be a partially covered triangle with level 1, which needs the three triangles with level 0 to be at least partially covered. But we have only four triangles partially covered or fully covered. This is impossible.

If $m = 2$, we can replace S by $V(G_{k,n}) \setminus S$ and find that it is just the situation $m = 1$, which is also impossible.

Case 2. $|S| < \frac{|V(G_{k,2})|}{2}$.

According to Lemma 3.4, we have $\frac{k-2}{2} \leqslant c_S(Q) \leqslant \frac{k-1}{2}$ for each partially covered sub-clique Q.

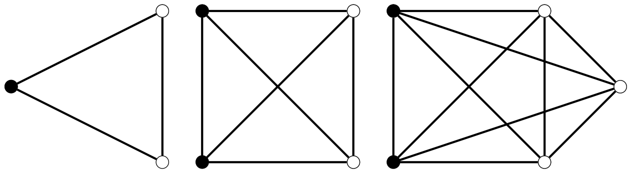
For the isoperimetric set S with minimum $p(S) > 0$ and $|S| < \frac{|V(G_{k,2})|}{2}$, and Q a partially covered sub-clique, we claim the following two properties:

(i) For each $v \in S \cap V(Q)$, we must have $f(v) \in S$ and $f^{-1}(v) \in S$. Otherwise, let $S' = S - v$. And we will get $|S'| = |S| - 1$ and $|\partial S'| \leqslant |\partial S| - k + 2c_S(Q)$, which means $i_{S'}(G_{k,n}) < i_S(G_{k,n})$ or $p(S') < p(S)$ or $c_{S'}(Q) < \frac{k-2}{2}$, a contradiction.

(ii) For each $u \in V(Q) \setminus S$, we must have $f(u) \notin S$ and $f^{-1}(u) \notin S$. Otherwise, let $S' = S + u$. And we will get $|S'| = |S| + 1 \leqslant \frac{|V(G_{k,2})|}{2}$ and $|\partial S'| \leqslant |\partial S| - 2c_S(Q) - 2 + k$, which means $i_{S'}(G_{k,n}) < i_S(G_{k,n})$, a contradiction.

Firstly, if k is odd, then we have $c_S(Q) = \frac{k-1}{2}$ and $i(G_{k,2}) \leqslant 1$. For $k \geqslant 7$, we have $m \geqslant 3$. Let $S' = S \setminus V(Q)$ for some Q. And we have $|S'| = |S| - c_S(Q)$ and $|\partial S'| \leqslant |\partial S| - (m-2)c_S(Q)$, yielding $i_{S'}(G_{k,2}) \leqslant i_S(G_{k,2})$, which means S is not an isoperimetric set of $G_{k,2}$ or $p(S') < p(S)$.

For $k = 5$, the partially covered sub-clique is given in Fig. 2(b). Since $|S| < (k-1)^2$, we can derive S' from S by covering the two uncovered vertices, and make $i_{S'}(G_{k,2}) < i_S(G_{k,2})$.



(a) $k = 4$. (b) $k = 5$. (c) $k = 6$.

Fig. 2. Possible partially covered sub-cliques (solid points are covered by S).

Secondly, if k is even, then we have $c_S(Q) = \frac{k-2}{2}$. For $k \geqslant 8$, we can construct S' by $S \setminus V(Q)$ for some partially covered Q, and get $|S'| = |S| - c_S(Q)$, $|\partial S'| \leqslant |\partial S| - 2c_S(Q)$, yielding $i_{S'}(G_{k,2}) < i_S(G_{k,2})$.

If $k = 4$, assume that Q_0 is a partially covered triangle with $L(Q_0) = 0$, $v \in S \cap V(Q_0)$, $v_1, v_2 \in V(Q_0) \setminus S$. Then $f(v) \in S \cap Q_1$ and $f^{-1}(v) \in S \cap Q_2$, where $L(Q_1) = L(Q_2) = L(f(v_1)) = L(f(v_2)) = L(f^{-1}(v_1)) = L(f^{-1}(v_2)) = 1$.

We claim that, between Q_1 and Q_2, there is at least one which is fully covered. Otherwise we could derive S' via $S \setminus (V(Q_0) \cup V(Q_1) \cup V(Q_2))$, and obtain $|S'| = |S| - 3$, $|\partial S'| = |\partial S| - 4$, which mean $i_{S'}(G_{k,2}) < i_S(G_{k,2})$ because $\frac{10}{9} < \frac{4}{3}$.

Moreover, $f(v_1), f^{-1}(v_1), f(v_2), f^{-1}(v_2)$ locate in three different triangles, which are at most partially covered. This contradicts there being only three triangles with the same level.

Finally, $k = 6$. If $p(S) = 1$, let $S' = S \cup V(Q)$ for the partially covered sub-clique Q, and make $|S'| = |S| + 3$, $|\partial S'| \leqslant |\partial S|$, which mean $i_{S'}(G_{k,2}) < i_S(G_{k,2})$. If $p(S) \geqslant 2$, assuming that Q_1 and Q_2 are two partially covered sub-cliques, by setting $S' = (S \cup V(Q_1)) \setminus V(Q_2)$ we will find $|S'| = |S| + 1$, $|\partial S'| \leqslant |\partial S| - 2$, which mean $i_{S'}(G_{k,2}) < i_S(G_{k,2})$.

Conclusively, all possible cases for S have been discussed and denied, which means minimum $p(S) > 0$ is impossible. We finish the proof. □

With Lemma 3.7, we are sure that $G_{k,2}$ has an isoperimetric set S with $p(S) = 0$. According to Proposition 2.6 and Lemma 3.3, we have

Theorem 3.8.

$$i(G_{k,2}) = \begin{cases} 1, & k \text{ is odd}, \\ \dfrac{(k-1)^2 + 1}{(k-1)^2}, & k \text{ is even}. \end{cases} \tag{3.4}$$

Now, the exact isoperimetric number of $G_{k,2}$ has been given. Generally, when $n \geqslant 3$, it is difficult to formulate the exact value of $i(G_{k,n})$, since it is much harder to prove the existence of isoperimetric set S with $p(S) = 0$, and also $i(G^*_{k,n})$ is not easy to obtain. However, we have the following upper bound:

Theorem 3.9.

$$i(G_{k,n}) \leqslant \frac{2}{n-1}.$$

Proof. For $n = 2$, we have presented the exact value of $i(G_{k,2})$, clearly satisfied.

When $n \geqslant 3$, let $S = \bigcup_{i=0}^{n-2} V_i$, where

$$V_i = \left\{ v^i = s_0 s_1 \cdots s_{i-1} \tilde{0} s_{i+1} \cdots s_{n-1} \mid s_j \in \{0, 1, \ldots, k-2\}, j \neq i \right\}, \quad i = 0, 1, \ldots, n-2.$$

We can easily verify

$$\begin{aligned} & g_j(v^i) \in V_i \subset S, \quad \forall\, v^i \in V_i, i = 0, 1, \ldots, n-2;\ j = 1, \ldots, k-2 \\ & f(v^i) \in V_{i-1} \subset S, \quad \forall\, v^i \in V_i, i = 1, \ldots, n-2 \\ & f^{-1}(v^i) \in V_{i+1} \subset S, \quad \forall\, v^i \in V_i, i = 0, 1, \ldots, n-3 \\ & f(v^0) \notin S, \quad \forall\, v^0 \in V_0 \\ & f^{-1}(v^{n-2}) \notin S, \quad \forall\, v^{n-2} \in V_{n-2}. \end{aligned}$$

Then $|S| = (n-1)(k-1)^{n-1}$ and $|\partial S| = 2(k-1)^{n-1}$. Therefore, $i(G_{k,n}) \leqslant i_S(G_{k,n}) = \frac{|\partial S|}{|S|} = \frac{2}{n-1}$. □

4. Concluding remarks

In this work, we present the exact isoperimetric number of the k-degree Cayley graph $G_{k,n}$ with $n = 2$, and also the general upper bound of $i(G_{k,n})$ is given.

On the basis of Lemma 3.7 and the constructive proof of Theorem 3.9, we could conjecture that Lemma 3.7 also holds with the general case when $n \geqslant 3$. Furthermore, we may get a better upper bound from more investigation on $G^*_{k,n}$.

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